

# On Fractional Quantum Hall Solitons and Chern-Simons Quiver Gauge Theories

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## Abstract

We investigate a class of hierarchical multiple layers of fractional quantum Hall soliton (FQHS) systems from Chern-Simons quivers embedded in M-theory on the cotangent on a 2-dimensional complex toric variety  $\mathbf{V}^2$ , which is dual to type IIA superstring on a 3-dimensional complex manifold  $\mathbf{CP}^1 \times \mathbf{V}^2$  fibered over a real line  $\mathbb{R}$ . Based on M-theory/Type IIA duality, FQHS systems can be derived from wrapped D4-branes on 2-cycles in  $\mathbf{CP}^1 \times \mathbf{V}^2$  type IIA geometry. In this realization, the magnetic source can be identified with gauge fields obtained from the decomposition of the R-R 3-form on a generic combination of 2-cycles. Using type IIA D-brane flux data, we compute the filling factors for models relying on  $\mathbf{CP}^2$  and the zeroth Hirzebruch surface.

**Keywords:** Quantum Hall Solitons, M-theory, Type IIA string, Chern-Simons Quivers.

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# 1 Introduction

Three dimensional Quantum Hall Systems (QHS) of condensed matter physics can be modeled using D-branes of type II superstrings [1, 2, 3]. The first ten dimensional superstring picture of Quantum Hall Effect (QHE) in (1+2)-dimensions has been given in terms of a stack of  $K$  D6-branes, a spherical D2-brane as well as dissolved D0-branes in D2 and a stack of  $N$  F-strings stretching between D2 and D6-branes [1]. The F-strings ending on the D2-brane have an interpretation in terms of the fractional quantum Hall particles and they are charged under the  $U(1)$  world-volume gauge field associated with the D0-branes. These solitonic objects behave as magnetic flux quanta dissolved in the D2-brane world-volume on which the QHS system reside. This string picture of QHE in (1+2)-dimensions has been extended to the compactification of type IIA superstring theory on the  $K3$  surface with singularities classified by Dynkin diagrams[4, 5, 6].

Alternatively, some efforts have been devoted to study fractional quantum Hall solitons (FQHS) in connections with (2+1)-dimensional Chern-Simons (CS) theory constructed by Aharony, Bergman, Jafferis and Maldacena. Recall that, ABJM theory is a 3-dimensional  $N = 6$  CS quiver with  $U(N)_k \times U(N)_{-k}$  gauge symmetry proposed to be dual to M-theory propagating on  $AdS_4 \times S^7/Z_k$ , with an appropriate amount of fluxes, or type IIA superstring on  $AdS_4 \times \mathbf{CP}^3$  for large  $k$  and  $N$  with  $k \geq N$  in the weakly interacting regime [7]. In the decoupling limit, the corresponding three dimensional conformal field theory (CFT<sub>3</sub>) is obtained from the physics of the multiple M2-branes placed at the orbifold space  $C^4/Z_k$ . It has been shown that QHE can be realized on the world-volume action of the M5-brane filling  $AdS_3$  inside  $AdS_4$ [8]. The model has been derived from  $d = 3$  flavored ABJM theory with the CS levels  $(1, -1)$ . FQHE systems can be also embedded in ABJM theory by adding fractional type IIA D-branes [9].

Recently, a possible extension of FQHS in ABJM theory has been given using type IIA dual geometry considered as the blown up of  $\mathbf{CP}^3$  by four-cycles which are isomorphic to  $\mathbf{CP}^2$ . Based on D6-branes wrapped 4-cycles and interacting with the R-R gauge fields in ABJM-like geometry, hierarchical multiple layers of FQHS systems have been given in [10].

Extended constructions can be characterized by a vector  $q_i$  and a real, symmetric and invertible symmetric matrix  $K_{ij}$ . These parameters classify various fractional quantum Hall (FQH) states. The choice of  $K_{ij}$  play important role in the embedding of CS in type II superstrings and M-theory compactified on deformed singular Calabi-Yau manifolds. In particular, the matrix  $K_{ij}$  can be identified with the intersection numbers of compact cycles in the internal space. These numbers are, up to some details, the opposite of the Cartan matrices of Lie algebras. This link may lead to a nice correspondence between FQHS models and 2-cycles involved in the deformation of toric singularities. More specifically, to each simple 2-cycle,

we associate a FQHS model.

In this work we discuss such CS realizations of QHS in 1+2 dimensions from M-theory on a real eight dimensional manifold. The manifold is realized explicitly as the cotangent bundle over a 2-dimensional complex toric variety  $\mathbf{V}^2$ . Up some details, the obtained models can be compared to Gaiotto-Witten theory describing  $N = 4$  CS superconformal theories. This connection can be obtained by imposing constraints on the possible values on CS levels satisfying the fundamental identity [11].

Since  $q_i$  and  $K_{ij}$  are connected via the filling factor, the embedding of CS in the string theory gives a toric geometric calculation for the filling factor. More precisely, starting from given two toric varieties describing quantum Hall systems and using CS gauge modeling a la Susskind, we determine the filling factors. In particular, based on M-theory/Type IIA duality, we give first Chern-Simons type theories describing 3-dimensional FQHS systems using D4-branes wrapping 2-cycles and interacting with the R-R gauge fields living on  $\mathbf{CP}^1 \times \mathbf{V}^2$  type IIA dual geometry. This class of QHS can be considered as a possible extension of the works given discussing the QHE in ABJM and its generalization [8, 9, 10]. This allows a geometric interpretation of filling factors in terms of the intersections between 2-cycles in  $H^2(\mathbf{V}^2, \mathbb{Z})$ . The matrix  $K_{ij}$  can be identified with the intersection matrix for curves embedded in  $\mathbf{CP}^1 \times \mathbf{V}^2$ . Then, we present explicit examples in order to illustrate the general idea. In particular, we discuss the case of  $\mathbf{CP}^2$  and the Hirzebruch surfaces.

The organization of this work is as follows. In the next section, we study FQHS systems in ABJM-like theory using D6-branes wrapping 4-cycles in  $\mathbf{CP}^3$ . In section 3, we derive a class of FQHS in (1+2)-dimensions using M-theory/Type IIA duality. In particular, we discuss FQHS model based on  $\mathbf{CP}^2$ , then we extend the analysis to the Hirzebruch surfaces in section 4. The last section is devoted to our conclusion.

## 2 QHS in ABJM theory

We start this section by recalling that the quantum Hall states are characterized by the filling factor  $\nu$  describing the ratio between the electronic density and the flux density. When  $\nu$  is a fractional value, it is called fractional quantum Hall effect (FQHE) for interacting electron systems. The first proposed series of the fractional quantum states was given by Laughlin and they are characterized by the filling factor  $\nu_L = \frac{1}{k}$  where  $k$  is an even integer for a boson electron and an odd integer for a fermionic electron [12]. At low energy, this model can be described by a 3-dimensional U(1) Chern-Simons theory coupled to an external electromagnetic field  $\tilde{A}$  with the following effective action

$$S_{CS} = -\frac{k}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge dA + \int_{\mathbb{R}^{1,2}} \frac{q}{2\pi} \tilde{A} \wedge dA, \quad (2.1)$$

where  $A$  is the dynamical gauge field,  $\tilde{A}$  is an external electromagnetic field, and where  $q$  is the charge of the electron [13, 14]. Extended models are characterized by a vector  $q_i$  and a real, symmetric and invertible symmetric matrix  $K_{ij}$ . These parameters play an important role in quiver construction of FQHE embedded in type II superstrings [4, 6] and M-theory compactified on 8-dimensional manifolds [5]. Following the Susskind approach for abelian field theory, these models can be described by the following action

$$S \sim \frac{1}{4\pi} \int_{\mathbb{R}^{1,2}} K_{ij} A^i \wedge dA^j + 2 \int_{\mathbb{R}^{1,2}} q_i \tilde{A} \wedge dA^i. \quad (2.2)$$

The external gauge field  $\tilde{A}$  couples now to each current  $\star dA^i$  with charge strengths  $eq_i$ . The  $K_{ij}$  matrix and the  $q_i$  charge vector in this effective field action suggests some physical concepts. Following the Wen-Zee model [14],  $K_{ij}$  and  $q_i$  are interpreted as order parameters and classify the various QHS states. Integrating over the all abelian gauge fields  $A^i$ , one gets the formulae for the filling factor of the system

$$\nu = q_i K_{ij}^{-1} q_j. \quad (2.3)$$

In what follows, we see that such CS quivers describing FQHS can be embedded either in ABJM theory with  $U(N)_k \times U(N)_{-k}$  gauge symmetry or more generally in toric CS quiver gauge theories. For simplicity, we first consider the case of the  $U(1)_k$  CS gauge theory in ABJM theory. Indeed, this model can be obtained from a D6-brane wrapping a four-cycle class  $[C_4]$  in  $H^4(\mathbf{CP}^3, \mathbb{Z})$  which is one-dimensional. On the gauge theory side of ABJM, the gauge symmetry  $U(N)_k \times U(N)_{-k}$  becomes  $U(N+M)_k \times U(N)_{-k}$ . To derive the first part of the action (2.1), we take just the  $U(1)$  abelian part of  $U(M)_k$  and assume that the remaining gauge factors are spectators. Indeed, on the D6-brane one can write down the following  $S_{WZ}$  action

$$S_{WZ} \sim T_6 \int_{\mathbb{R}^{1,6}} F \wedge F \wedge A_3^{RR}, \quad (2.4)$$

where  $T_6$  is the D6-brane tension and where the gauge field  $A_3^{RR}$  is the R-R 3-form coupled to the D2-brane of type IIA superstring. Integrating by parts and integrating the result over the 4-cycle  $C_4$ , one gets

$$- \frac{k}{4\pi} \int_{\mathbb{R}^{1,2}} A \wedge F \quad (2.5)$$

where  $k = \frac{1}{2\pi} \int_{C_4} (dA_3^{RR})$  is produced now by  $k$  D4-flux. To couple the system to an external gauge field, we need to turn on the RR 5-form  $A_5^{RR}$  which is coupled to the D4-brane. Decomposing this gauge field as follows

$$A_5^{RR} \rightarrow \tilde{A} \wedge \omega \quad (2.6)$$

where  $\omega$  is a harmonic 4-form dual to the four-cycle  $C_4$ , the WZ action  $\int_{\mathbb{R}^{1,6}} A_5^{RR} \wedge F$  on a D6-brane gives the second term of the action (2.1), namely,

$$q \int_{\mathbb{R}^{1,2}} \tilde{A} \wedge F \quad (2.7)$$

where  $q = \int_{C_4} \omega$  and  $\tilde{A}$  is an U(1) gauge field which serves as the external gauge field that couples to the gauge fields living on the D6-brane world-volume. The above effective action on the D6-brane world-volume reproduces the following filling factor

$$\nu = \frac{q^2}{k}. \quad (2.8)$$

From this equation, it follows that the filling factor depends on the D4-branes and the harmonic 4-forms defined on  $\mathbf{CP}^3$ . It turns out that known values could be obtained by taking particular choices of such parameters. Moreover, these FQHS in  $\text{AdS}_4/\text{CFT}_3$  have been extended to models based on IIA dual geometry realized as the blown up of  $\mathbf{CP}^3$  by four-cycles which are isomorphic to  $\mathbf{CP}^2$ . In particular, we proposed a stringy hierarchical description of multi-layer systems in terms of wrapped D6-branes on the blown up four-cycles [10].

In what follows, we investigate FQHS system embedded in CS quiver gauge theories arising from M-theory on the cotangent bundle on a two dimensional complex toric variety  $\mathbf{V}^2$ . More precisely, we discuss the case of two dimensional complex projective space and the zeroth Hirzebruch surface.

### 3 FQHS and Chern-Simons quivers

Many models have been given to extend ABJM theory. They are conjectured to be gauge field duals of  $\text{AdS}_4$  background in type IIA superstring and M-theory compactifications on eight dimensional manifolds. In particular, these kinds of CS quiver theories can be obtained in terms of M2-branes placed at hyper toric singularities of eight-dimensional manifolds. Recall that, the simple model is described by abelian gauge factors  $\text{U}(1)_{k_1} \times \text{U}(1)_{k_2} \times \dots \times \text{U}(1)_{k_n}$ , where  $k_i$  denote the Chern-Simons levels for each abelian factor U(1). Geometrically, these models can be encoded in a quiver formed by  $n$  nodes where each factor U(1) is associated with a node while the matter is represented by the link between nodes [15, 16]. For these models, it has been shown that Chern-Simons levels  $k_i$  are subject to

$$\sum_i k_i = 0. \quad (3.1)$$

For the  $\prod_i \text{U}(N_i)_{k_i}$  non abelian gauge group, the Chern-Simons levels  $k_i$  and the set of the ranks of the gauge groups  $N_i$  are constrained by

$$\sum_i k_i N_i = 0. \quad (3.2)$$

The CS quivers we consider here are obtained from M-theory on the cotangent bundle over a two dimensional toric variety  $\mathbf{V}^2$ . This internal space is built in terms of a bi-dimensional  $\text{U}(1)^r$  sigma model with eight supercharges ( $\mathcal{N} = 4$ ) and  $r + 2$  hypermultiplets. It has been considered to be the solution of the following D-flatness condition

$$\sum_{i=1}^{r+2} Q_i^a [\phi_i^\alpha \bar{\phi}_{i\beta} + \phi_i^\beta \bar{\phi}_{i\alpha}] = \vec{\xi}_a^\alpha \vec{\sigma}_\beta^\alpha, \quad a = 1, \dots, r \quad (3.3)$$

where  $Q_i^a$  is a matrix charge specified later on.  $\phi_i^\alpha$ 's ( $\alpha = 1, 2$ ) denote the component field doublets of each hypermultiplets ( $i = 1 \dots, r + 2$ ).  $\vec{\xi}_a$  are the Fayet-Iliopoulos (FI) 3-vector couplings rotated by the  $\text{SU}(2)$  symmetry, and  $\vec{\sigma}_\beta^\alpha$  are the traceless  $2 \times 2$  Pauli matrices. The explicit solution depends on the values of the FI couplings. For a particular region in the moduli space where  $\vec{\xi}_a^1 = \vec{\xi}_a^2 = 0$  and  $\vec{\xi}_a^3 > 0$ , (3.3) describes the cotangent bundle over a complex two-dimensional toric variety  $\mathbf{V}^2$  defined by

$$\sum_{i=1}^{2+r} Q_i^a |\phi_i^1|^2 = \vec{\xi}_a^3, \quad a = 1, \dots, r. \quad (3.4)$$

Using M-theory/type IIA duality, it follows that M theory compactified on such geometries is dual to type IIA superstring on 3-dimensional complex manifolds  $X_3$  fibered over a real line  $R$  with D6-branes. In fact, one can show that  $X_3$  can be described as a  $\mathbf{CP}^1$  fibration over the base  $\mathbf{V}^2$ . Instead of being general, let us consider first a toy model where  $\mathbf{V}^2$  is  $\mathbf{CP}^2$  associated with  $r = 1$  and the vector charge  $Q_i = (1, 1, 1)$ . The corresponding quiver has 3 nodes, where each node is associated with an  $\text{U}(N)$  gauge symmetry factor. For this CS quiver, we have a level vector  $(k_1, k_2, k_3)$  such that

$$k_1 + k_2 + k_3 = 0. \quad (3.5)$$

It turns out that this model could be related to ABJM theory by adding D6-branes wrapping 4-cycles of the blown up  $\mathbf{CP}^3$  by  $\mathbf{CP}^2$  at singular toric points. In this way of thinking and solving the constraints (3.2), the gauge symmetry  $\text{U}(N)_k \times \text{U}(N)_{-k}$  of ABJM theory can change into  $\text{U}(N)_k \times \text{U}(N)_k \times \text{U}(N)_{-2k}$ . The IIA/M-theory duality predicts that this gauge theory should be dual to type IIA superstring on  $AdS_4 \times \mathbf{CP}^1 \times \mathbf{CP}^2$ . Roughly speaking, the CS gauge theory describing FQHS model can be obtained from D4-branes moving on

$\mathbf{CP}^1 \times \mathbf{CP}^2$ . In type IIA geometry, D4-branes can wrap  $\mathbf{CP}^1$  and a particular complex curve class  $[C]$  in  $H^2(\mathbf{CP}^2, \mathbb{Z})$ . On the gauge theory side, the gauge symmetry becomes  $U(N+M)_k \times U(N)_k \times U(N+M')_{-2k}$ . The constraint (3.2) requires that

$$M = 2N \quad M' = N. \quad (3.6)$$

As before, the CS quiver theory describing our FQHS system will be in the  $U(2N)_k \times U(N)_{-2k}$  part of  $U(3N)_k \times U(N)_k \times U(2N)_{-2k}$ . Extracting an  $U(2) \times U(1)$ , we can obtain the FQH field theory from D4-branes wrapping individually  $\mathbf{CP}^1$  and  $C$ . To see that let us consider first the case of the abelian part  $U(1)$  corresponding to  $C$ . Indeed, on the 5-dimensional world-volume of each D4-brane we have  $U(1)$  gauge symmetry. The corresponding effective action can take the following form

$$S_{D4} \sim T_4 \int_{\mathbb{R}^{1,2} \times C} d^5 \sigma e^{-\phi} \sqrt{-\det(G + 2\pi F)} + T_4 \int_{\mathbb{R}^{1,2} \times C} F \wedge F \wedge A_1^{RR} \quad (3.7)$$

where  $T_4$  is the brane tension and where the gauge field  $A_1^{RR}$  is the R-R 1-form coupled to the D0-brane of type IIA superstring. Ignoring the first term and integrating by part, the WZ action on the D4-brane world-volume becomes

$$\int_{\mathbb{R}^{1,2} \times C} F \wedge F \wedge A_1^{RR} = - \int_{\mathbb{R}^{1,2} \times C} A \wedge F \wedge (dA_1^{RR}) \quad (3.8)$$

Then, we get the well known Chern-Simons term

$$S_{WZ} \sim \int_{\mathbb{R}^{1,2}} A \wedge F. \quad (3.9)$$

The presence of the R-R gauge field sourced by the anti-D6-flux leads to  $-2k = \frac{1}{2\pi} \int_C (dA_1^{RR})$ . To couple the system to an external gauge field, one needs an extra D4-brane wrapping the cycle  $C$  in the presence of the RR 3-form  $A_3^{RR}$  which is sourced by a D2-brane dual to a D4-brane. After the compactification, this gauge field decomposes into

$$A_3^{RR} \rightarrow \tilde{A} \wedge \omega \quad (3.10)$$

where  $\omega$  is a harmonic 2-form on  $C$ . In this way, the WZ action  $\int A_3^{RR} \wedge F$  on a D4-brane wrapping  $C$  gives

$$q \int_{\mathbb{R}^{1,2}} \tilde{A} \wedge F \quad (3.11)$$

where  $\tilde{A}$  is the  $U(1)$  gauge field which can be obtained from the dimensional reduction of the RR 3-form  $A_3^{RR}$ . This  $U(1)$  gauge field can be interpreted as a magnetic external gauge field that couples to our QHS. We can follow the same steps to construct a non abelian

effective Chern-Simons gauge theory with  $U(2)$  gauge fields. Roughly speaking, thanks to  $\nu = \nu_1(U(2)) + \nu_2(U(1))$ , and putting the charge like  $q_i = (1, 1, 1)$ , the corresponding filling factor reads as

$$\nu = \frac{2}{k} - \frac{1}{2k} = \frac{3}{2k}. \quad (3.12)$$

## 4 FQHS in CS quivers on Hirzebruch surfaces

Let us extend the result obtained in previous sections to quivers with more than three nodes. There are various ways of doing that. One possibility can be realized by replacing two dimensional projective spaces either by the Hirzebruch surfaces or the del Pezzo surfaces. The corresponding CS quivers involve more than three  $U(N_i)$  gauge symmetry factors. The general study is beyond the scope of the present work, though we will consider an explicit example corresponding to the zeroth Hirzebruch surfaces  $F_0$ . Other examples may be dealt with in a similar manner. We will briefly comment on various simple extension in the conclusion.

Recall from literature that  $F_0$  is a two-dimensional toric surface defined by a trivial fibration of  $\mathbf{CP}^1$  over  $\mathbf{CP}^1$ . In  $N = 2$  sigma model language,  $F_0$  is realized as the target space  $U(1) \times U(1)$  gauge theory with four chiral fields with charges  $(1, 1, 0, 0)$  and  $(0, 0, 1, 1)$ . As proposed in [16], the corresponding CS quiver is given by  $U(N) \times U(N) \times U(N) \times U(N)$  where the CS levels are  $(k, -k, k, -k)$ . This model corresponds to a circular quiver with four nodes. The FQHS system we are interested in appears as a CS quiver model obtained from gauge theories living on the world-volume of the wrapped D4-branes on 2-cycles in type IIA geometry which is given by the trivial fibration of  $\mathbf{CP}^1$  over  $F_0$ . Indeed, there are three complex curves on which the D4-branes can be wrapped. D4-branes can wrap  $\mathbf{CP}^1$  and two complex curves class  $[C_i]$  in  $H^2(F_0, \mathbb{Z})$ . In this realization of FQHS, the fractional D4-branes will give three gauge factors. Indeed, consider three stacks  $(M_1, M_2, M_3)$  of D4-branes. On the IIA gauge theory side, the gauge symmetry becomes  $U(N + M_1)_k \times U(N)_{-k} \times U(N + M_2)_{-k} \times U(N + M_3)_k$ . A particular solution of (3.2) requires that

$$M_1 = N, \quad M_2 = 2N, \quad M_3 = N. \quad (4.1)$$

Our CS quiver theory describing FQHS systems will be in the  $U(N)_k \times U(2N)_{-k} \times U(N)_k$  part. As before, thanks to  $\nu = \nu_1(U(1) \times U(1)) + \nu_2(U(2))$ , we can find the the filling factor of the model. Indeed, consider first the bi-layer system corresponding to  $U(1) \times U(1)$  quiver



gauge theory. The corresponding matrix  $K_{ij}$  takes the form

$$K_{ij}(\mathrm{U}(1) \times \mathrm{U}(1)) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

where the integer  $k$  can be interpreted geometrically in terms of the self intersection of complex curves with genus  $g > 0$ . From type IIA string point of view, this model can be obtained from two D4-branes wrapping two identical complex curves. Evaluating (2.3) for the charges  $q_i = (q_1, q_2)$  yields

$$\nu_1 = \frac{q_1^2 + q_2^2}{k}. \quad (4.2)$$

This is a general expression depending on the vector charge. Particular choice of the D-brane flux and the charge vector may recover some observed values of the filling factor. Adding now the non abelian contributions, the total filling factor can be written as

$$\nu = \frac{q_1^2 + q_2^2 - 2q_3^2}{k}. \quad (4.3)$$

It is worth noting that the vanishing filling factor behavior has to do with the condition  $q_1^2 + q_2^2 = 2q_3^2$ . This can be fixed by the flux on the world-volume D6-branes. The Hall conductivity is quantized in terms of the CS level being identified with the D6-flux and the charges given in terms of the integral of harmonic 2-forms over dual 2-cycles embedded in type IIA geometry.

## 5 Conclusion

In this letter, we have studied a class of FQHS in toric CS quivers from M-theory on the contangent bundle over a two dimensional toric variety  $\mathbf{V}^2$ . The dual type IIA geometries are realized as  $AdS_4 \times \mathbf{CP}^1 \times \mathbf{V}^2$ . Using M-theory/Type IIA duality, we have presented hierarchical stringy descriptions using quiver gauge models living on stacks of D4-branes wrapping 2-cycles in  $\mathbf{CP}^1 \times \mathbf{V}^2$  and interacting with R-R gauge fields. In particular, we have given two simple examples of FQHS systems. The first model is based on the CS quiver relying on  $\mathbf{CP}^1 \times \mathbf{CP}^2$ , while the second one is associated with quiver gauge theory describing system based on D4-branes which wrap 2-cycles inside  $\mathbf{CP}^1 \times F_0$ . This analysis can be adapted to other multilayered systems by considering more general two dimensional toric manifolds. The manifolds are given by  $\mathbf{C}^{r+2}/\mathbf{C}^{*r}$ , where the  $\mathbf{C}^{*r}$  actions are given by  $z_i \rightarrow \lambda^{q_i^a} z_i$ ,  $i = 1, \dots, r+2; a = 1, \dots, r$ . Clearly, type IIA geometry will have  $h^{1,1} = r+1$ . The right interpretation of this is that we have  $r+1$  2-cycles associated with wrapped D4-

branes. The corresponding FQHS model involves  $r + 1$  gauge factors. In connection with that, it would therefore be of interest to consider such systems using tools developed in toric geometry [17, 18]. It is natural to consider del Pezzo surfaces. Other possibilities for  $\mathbf{V}^2$  are local ALE spaces with ADE singularities.

On the other hand, motivated by supersymmetric part of ABJM theory and supersymmetric QHE, it should be interesting to study such realizations in terms of M-theory on eight dimensional spaces. We hope, all these open questions will be addressed in future works.

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